LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – MATHEMATICS

FIRST SEMESTER – NOVEMBER 2007

MT 1804 - LINEAR ALGEBRA

AB 19

Date : 25/10/2007 Dept. No. Max. : 100 Marks

Time : 1:00 - 4:00

Answer ALL Questions.

I. a) i) Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$. Prove that T is diagonalizable by exhibiting a basis of R^3, each vector of which is a characteristic vector of T.

Or

ii) Let T be a linear operator on a finite dimensional vector space V. Let $c_1, c_2 \ldots c_k$ be the distinct characteristic values of T and let W_i be the null space of (T- c_i I). If the characteristic polynomial for T is f= $(x-c_1)^{d_1}(x-c_2)^{d_2}\ldots(x-c_k)^{d_k}$, dim W_i =d$_i$, i=1, 2… k then prove that dim V= $\dim W_1 + \dim W_2 + \ldots + \dim W_k$ (5)

b) i) Let T be a linear operator on a finite dimensional vector space V. Prove that the minimal polynomial for T divides the characteristic polynomial for T.

Or

ii) Let V be a finite dimensional vector space over F and T be a linear operator on V then prove that T is diagonalizable if and only if the minimal polynomial for T has the form p= $(x-c_1)(x-c_2)\ldots(x-c_k)$ where $c_1, c_2 \ldots c_k$ are distinct elements of F. (15)

II. a) i) Define independent subspaces of a vector space V.

ii) Let V be a finite dimensional vector space. Let $W_1, W_2, \ldots W_k$ be independent . subspaces of V and let $W = W_1 + W_2 + \ldots W_k$. Then prove that for each j, $2 \le j \le k$, $W_j \cap (W_1 + W_2 + \ldots + W_{j-1}) = \{0\}$ (2+3)

Or

iii) Let T be a linear operator on a finite dimensional vector space V and let $E_1, E_2, \ldots E_k$ are linear operators on V such that 1) each E_i is a projection 2) $E_i E_j = 0$ if i ≠ j 3) $I = E_1 + E_2 + \ldots E_k$ and let W_i is the range of E_i. If each W_i is invariant under T then prove that T $E_i = E_i$ T, i=1,2 .. k (5)

b) i) State and prove primary decomposition theorem.

Or

ii) Let α be any non-zero vector in V and p$_\alpha$ be the T-annihilator of α. Then prove that if deg p$_\alpha$ =k and if U is the linear operator on Z(α;T) induced by T, then k=dim Z(α;T) and the minimal polynomial for U is p$_\alpha$. (15)

III. a) i) If T is a linear operator on a finite dimensional vector space V, then prove that every T-admissible subspace has a complementary subspace which is also invariant under T.

Or

ii) Let A= $\begin{pmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{pmatrix}$ be the complex 3X3 matrix. Prove that A is similar to diagonal matrix if and only if a=0. (5)

b) i) State and prove Generalized Cayley-Hamilton theorem.

Or

ii) Let T be a linear operator on a finite dimensional vector space V and let W_0 be a proper T-admissible subspace of V. Then prove that there exist non-zero vectors α_1, α_2 ... α_r in V with respective T-annihilators $p_1, p_2, ...p_r$ such that V=$W_0 \oplus Z(\alpha_1;T) \oplus Z(\alpha_2;T) .... \oplus Z(\alpha_r;T)$ and p_k divides p_{k-1}, k=2,3 … r. (15)

IV. a) i) Let V be a finite dimensional inner product space and f a form on V. Then prove that there is a unique operator T on V such that f (α, β)=$(T\alpha / \beta)$ for all α, β in V where (α / β) is the inner product defined on V.

Or

ii) Let V be a complex vector space and f a form on V. Then prove that f is hermitian if and only if f (α, α) is real for every α. (5)

b) i) Let A=$\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$ and let g be the form defined by g(X,Y)=$Y^* AX$. Is g an inner product?

ii) State and prove Principle Axis theorem. (6+9)

Or

iii) Let f be a form on a real or complex vector space V and $\{\alpha_1, \alpha_2, ..., \alpha_r\}$ a basis for the finite dimensional subspace W of V. Let M be the rxr matrix with entries $M_{jk} = f(\alpha_k, \alpha_j)$ and W' the set of all vectors β in V such that f (α, β)=0 for all $\alpha \in$ W. Then prove that W' is a subspace of V and $W \cap W' = \{0\}$ if and only if M is invertible and when this is the case V=W+W'. (15)

V. a) i) Let V be a vector space over the field F. Define a bilinear form f on V and prove that the function defined by f (α, β) =$L_1(\alpha) L_2(\alpha)$ is bilinear.

Or

ii) State and prove polarization identity for symmetric bilinear form f. (5)

b) i) Let V be a finite dimensional vector space over the field of complex numbers. Let f be a symmetric bilinear form on V which has rank r. Then prove that there is an ordered basis $B = \{\beta_1, \beta_2, ...\beta_n\}$ for V such that the matrix of f in the ordered basis B is diagonal and f (β_i, β_j) = $\begin{cases} 1, & \text{j=1,2,..r} \\ 0, & \text{j>r} \end{cases}$

Or

ii) If f is a non-zero skew-symmetric bilinear form on a finite dimensional vector space V then prove that there exist a finite sequence of pairs of vectors, $(\alpha_1, \beta_1), (\alpha_2, \beta_2), ...(\alpha_k, \beta_k)$ with the following properties:

1) f (α_j, β_j)=1, j=1,2,,…,k.

2) f (α_i, α_j)=f (β_i, β_j)=f (α_i, β_j)=0, $i \neq j$.

3) If W_j is the two dimensional subspace spanned by α_j and β_j, then V=$W_1 \oplus W_2 \oplus ... W_k \oplus W_0$ where W_0 is orthogonal to all α_j and β_j and the restriction of f to W_0 is the zero form. (15)

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